

Universal Functions

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Abstract¹

A function of two variables $F(x, y)$ is universal iff for every other function $G(x, y)$ there exists functions $h(x)$ and $k(y)$ with

$$G(x, y) = F(h(x), k(y)).$$

Sierpinski showed that assuming the continuum hypothesis there exists a Borel function $F(x, y)$ which is universal. Assuming Martin's Axiom there is a universal function of Baire class 2. A universal function cannot be of Baire class 1. Here we show that it is consistent that for each α with $2 < \alpha < \omega_1$ there is a universal function of class α but none of class $\beta < \alpha$. We show that it is consistent with ZFC that there is no universal function (Borel or not) on the reals, and we show that it is consistent that there is a universal function but no Borel universal function. We also prove some results concerning higher

¹ Mathematics Subject Classification 2000: 03E15 03E35 0350

Keywords: Borel function, Universal, Martin's Axiom, Baire class, cardinality of the continuum, Cohen real model.

Results obtained Mar-Jun 2009, Nov 2010. Last revised April 2012.

arity universal functions. For example, the existence of an F such that for every G there are h_1, h_2, h_3 such that for all x, y, z

$$G(x, y, z) = F(h_1(x), h_2(y), h_3(z))$$

is equivalent to the existence of a 2-ary universal F , however the existence of an F such that for every G there are h_1, h_2, h_3 such that for all x, y, z

$$G(x, y, z) = F(h_1(x, y), h_2(x, z), h_3(y, z))$$

follows from a 2-ary universal F but is strictly weaker.

1 Introduction

Definition 1.1 *A function $F : X \times X \rightarrow X$ is universal iff for any*

$$G : X \times X \rightarrow X$$

there is $g : X \rightarrow X$ such that for all $x, y \in X$

$$G(x, y) = F(g(x), g(y)).$$

Sierpinski asked² when X is the real line if there always is a Borel function which is universal. He had shown that there is a Borel universal function assuming the continuum hypothesis (Sierpinski [20]).

Without loss we may use different functions on the x and y coordinates, i.e., $G(x, y) = F(g_0(x), g_1(y))$ in the definition of universal function F . To see this suppose we are given F^* such that for every G we may find g_0, g_1 with $G(x, y) = F^*(g_0(x), g_1(y))$ for all x, y . Then we can construct a universal F which uses only a single g . Take a bijection, i.e., pairing function between $X \times X$ and X , i.e., $(x_0, x_1) \mapsto \langle x_0, x_1 \rangle$. Define

$$F(\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle) = F^*(x_0, y_1)$$

where $\langle x_0, x_1 \rangle$ is a pairing function. Given any g_0, g_1 define

$$g(u) = \langle g_0(u), g_1(u) \rangle$$

²Scottish book, Mauldin [9] problem 132.

and note that

$$F(g(x), g(y)) = F^*(g_0(x), g_1(y))$$

for every x, y .

In the case $X = 2^\omega$ there is a pairing function which is a homeomorphism and hence the Borel complexity of F and F^* are the same. For abstract universal F a pairing function exists for any infinite X by the axiom of choice.

In section 2 we show that the existence of a Borel universal functions is equivalent to under a weak cardinality assumption to the statement that every subset of the plane is in the σ -algebra generated by the abstract rectangles. We also show that a universal function cannot be of Baire class 1.

In section 3 we prove some results concerning Martin's axiom and universal function. We show that although MA implies that there is a universal function of Baire class 2 it is consistent to have MA_{\aleph_1} hold but no Borel universal functions.

In section 4 we consider universal functions of a special kind. For example, $F(x, y) = k(x + y)$. We also discuss special versions due to Todorcevic and Davies.

In section 5 we consider abstract universal functions, i.e., those defined on a cardinal κ with no notion of definability, Borel or otherwise. We show that if $2^{<\kappa} = \kappa$, then they exists. We also show that it is consistent that none exists for κ equal to the continuum. We also prove some weak abstract versions of universal functions from the assumption MA_{\aleph_1} .

In section 6 we take up the problem of universal functions of higher arity. We show that there is a natural hierarchy of such notions and we show that this hierarchy is strictly descending.

2 Borel Universal Functions

Definition 2.1 We let \mathcal{R} denote the family of abstract rectangles,

$$\mathcal{R} = \{A \times B : A, B \subseteq 2^\omega\}.$$

Definition 2.2 $\Sigma_\alpha^0(\mathcal{R})$ and $\Pi_\alpha^0(\mathcal{R})$ for $\alpha < \omega_1$ are inductively defined by:

- $\Sigma_0^0(\mathcal{R}) = \Pi_0^0(\mathcal{R}) =$ the finite boolean combinations of sets from \mathcal{R} ,
- $\Sigma_\alpha^0(\mathcal{R})$ is the countable unions of sets from $\Pi_{<\alpha}^0(\mathcal{R}) = \bigcup_{\beta < \alpha} \Pi_\beta^0(\mathcal{R})$,
and

- $\Pi_\alpha^0(\mathcal{R})$ is the countable intersections of sets from $\Sigma_{<\alpha}^0(\mathcal{R})$.

Definition 2.3 A Borel function $F : 2^\omega \times 2^\omega \rightarrow 2^\omega$ is at the α -level iff for any n the set $\{(u, v) : F(u, v)(n) = 1\}$ is Σ_α^0 .

We remark that a Borel function at level α is in Baire class α , but not the converse. In the context of 2^ω a function is of Baire class α iff the preimage of every clopen set is $\Delta_{\alpha+1}$. For more on the classical theory of Baire class α , see Kechris [4] p. 190.

Theorem 2.4 Suppose that $2^{<\mathfrak{c}} = \mathfrak{c}$, then the following are equivalent:

1. There is a Borel function $F : 2^\omega \times 2^\omega \rightarrow 2^\omega$ which is universal.
2. Every subset of the plane $2^\omega \times 2^\omega$ is in the σ -algebra generated by the abstract rectangles, \mathcal{R} .

Furthermore, $\mathcal{P}(2^\omega \times 2^\omega) = \Sigma_\alpha^0(\mathcal{R})$ iff F can be taken to be the α -level.

Proof

(1) \rightarrow (2).

Suppose there is a Borel universal $F : 2^\omega \times 2^\omega \rightarrow 2$. Let $A \subseteq 2^\omega \times 2^\omega$ be arbitrary and suppose $g : 2^\omega \rightarrow 2^\omega$ has the property that

$$\forall x, y \quad (x, y) \in A \text{ iff } F(g(x), g(y)) = 1.$$

Let B be the Borel set $F^{-1}(1)$. Then B is generated by countable unions and intersections from sets of the form $C \times D$, for C, D clopen subsets 2^ω . Note that $(x, y) \in A$ iff $(g(x), g(y)) \in B$. Define $h(x, y) = (g(x), g(y))$ and note that

$$h^{-1}(C \times D) = g^{-1}(C) \times g^{-1}(D)$$

for all sets $C, D \subseteq 2^\omega$. Since

$$A = h^{-1}(B),$$

and since preimages pass over countable unions and intersections it follows that A is in the σ -algebra of abstract rectangles. Furthermore if B is Σ_α^0 , then A is $\Sigma_\alpha^0(\mathcal{R})$.

(2) \rightarrow (1).

We show first that there exists an $X \subseteq 2^\omega$ of cardinality \mathfrak{c} which has the property that every $Y \subseteq X$ of cardinality strictly smaller than \mathfrak{c} is Borel relative to X , i.e., is the intersection of a Borel set with X . See Bing, Bledsoe, and Mauldin [2]. Let $A \subseteq \mathfrak{c} \times \mathfrak{c}$ be such that for every $B \in [\mathfrak{c}]^{<\mathfrak{c}}$ there exists $\alpha < \mathfrak{c}$ such that

$$B = A_\alpha =^{def} \{\beta : (\alpha, \beta) \in A\}.$$

This is possible because $2^{<\mathfrak{c}} = \mathfrak{c}$. Since A is in the σ -algebra generated by the abstract rectangles, there exists a sequence $A_n \subseteq \mathfrak{c}$ for $n < \omega$ such that A is in the σ -algebra generated by $\{A_n \times A_m : n, m < \omega\}$. Let $f : \mathfrak{c} \rightarrow 2^\omega$ be the Marczewski characteristic function for the sequence $(A_n : n < \omega)$, i.e.,

$$f(x)(n) = \begin{cases} 0 & \text{if } x \notin A_n \\ 1 & \text{if } x \in A_n \end{cases}$$

Let $X = f(\mathfrak{c})$. Let us check that X has the required property. Let Y be a subset of X of cardinality less than \mathfrak{c} , and let B be a subset of \mathfrak{c} of cardinality less than \mathfrak{c} such that $Y = f(B)$. Each set of the form $A_n \times A_m$ is the preimage under f of a clopen subset of $2^\omega \times 2^\omega$. Again using the fact that preimages pass over countable unions and intersections, we can find a Borel subset $2^\omega \times 2^\omega$ whose preimage under f is A . Then Y will be one section of this set, intersected with X . Also note that if A is $\Sigma_\alpha^0(\mathcal{R})$, then every subset of X of cardinality strictly smaller than \mathfrak{c} is Σ_α^0 relative to X .

Now let $U \subseteq 2^\omega \times 2^\omega$ be a universal Σ_α^0 set. Define $G : 2^\omega \times 2^\omega \rightarrow 2^\omega$ by

$$\forall n \quad (G(x, y)(n) = 1 \text{ iff } (x_n, y) \in U)$$

where $x \mapsto (x_n : n < \omega) \in (2^\omega)^\omega$ is a homeomorphism.

Let $f_1 : \mathfrak{c}^2 \rightarrow 2^\omega$ be an arbitrary function with the property that $\alpha > \beta \rightarrow f_1(\alpha, \beta) = \vec{0}$ (the identically zero map). We claim that there exists $h_1, h_2 : \mathfrak{c} \rightarrow 2^\omega$ such that

$$f_1(\alpha, \beta) = G(h_1(\beta), h_2(\alpha)) \text{ for all } (\alpha, \beta) \in \mathfrak{c}^2.$$

To see this, let $X = \{x_\gamma : \gamma < \mathfrak{c}\}$. Let $h_2(\alpha) = x_\alpha$. For each n and β note that

$$B_n =^{def} \{x_\alpha : f_1(\alpha, \beta)(n) = 1\}$$

is a subset of X of cardinality less than \mathfrak{c} and so there exists $y_n \in 2^\omega$ such that $B_n = X \cap U_{y_n}$. Construct $h_1(\beta) = y$ corresponding to such a sequence $(y_n : n < \omega)$.

By an analogous argument, if $f_2 : \mathfrak{c}^2 \rightarrow 2^\omega$ is an arbitrary map with the property that $\beta > \alpha \rightarrow f_2(\alpha, \beta) = \vec{0}$, then there exists $k_1, k_2 : \mathfrak{c} \rightarrow 2^\omega$ such that

$$f_2(\alpha, \beta) = G(k_1(\alpha), k_2(\beta)) \text{ for all } (\alpha, \beta) \in Q_2.$$

Now define $F : 2^\omega \times 2^\omega \rightarrow 2^\omega$ by:

$$F((x_1, y_1), (x_2, y_2)) = \max(G(x_2, x_1), G(y_1, y_2))$$

where $\max : 2^\omega \times 2^\omega \rightarrow 2^\omega$ is the pointwise maximum, i.e., $\max(u, v) = w$ iff $w(n)$ is the maximum of $u(n)$ and $v(n)$ for each $n < \omega$. Then

$$F((x_1, y_1), (x_2, y_2))(n) = 1 \text{ iff } G(x_1, x_2)(n) = 1 \text{ or } G(y_2, y_1)(n) = 1.$$

We show the F is universal. Given an arbitrary $f : \mathfrak{c} \times \mathfrak{c} \rightarrow 2^\omega$ we can find f_1 and f_2 as above so that

$$f(\alpha, \beta) = \max(f_1(\alpha, \beta), f_2(\alpha, \beta)) \text{ for all } (\alpha, \beta) \in \mathfrak{c}^2.$$

Define $l_1(\alpha) = (h_2(\alpha), k_1(\alpha))$ and $l_2(\beta) = (h_1(\beta), k_2(\beta))$. Then

$$f(\alpha, \beta) = F(l_1(\alpha), l_2(\beta)) \text{ for all } \alpha, \beta < \mathfrak{c}.$$

Also F is at the α -level, i.e., for any n the set $\{(u, v) : F(u, v)(n) = 1\}$ is Σ_α^0 .

QED

Corollary 2.5 *It is consistent that for each α with $2 < \alpha < \omega_1$ there is a universal function of Baire class α but none of class $\beta < \alpha$. It is consistent that there is a universal function but no Borel universal function. If $\mathfrak{p} = \mathfrak{c}$, then there is a universal function of Baire class 2.*

Proof

This follows from corresponding results about the σ -algebra of abstract rectangles, see Miller [10] Theorem 37. The existence of an abstract universal function follows from $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ Theorem 5.1 and this holds in many models in which not every subset of the plane is in the σ -algebra generated by the abstract rectangles. For example, Kunen in his thesis showed this is true in the Cohen real model. The cardinal \mathfrak{p} is the psuedo-intersection number. An equivalent definition for it is the smallest cardinal for which Martin's Axiom of for σ -centered posets fails. This is due to Bell [1]), for the proof Bell's

Theorem see also Weiss [22]. Proposition 6.15 shows that if $\mathfrak{p} = \mathfrak{c}$, then there is a universal function of Baire class 2.

QED

Question 2.6 *Suppose there is a universal function of Baire class α . Then is there a universal function of level α ?*

Proposition 2.7 *A universal function cannot be of Baire class 1.*

Proof

Suppose that F is of Baire class 1. Let $\{h_\xi\}_{\xi \in \mathfrak{c}}$ enumerate all functions from a countable subset of 2^ω whose range is dense in itself. Let $\{r_\xi\}_{\xi \in \mathfrak{c}}$ enumerate all 2^ω . For each ξ partition the domain of h_ξ into A_ξ and B_ξ such that

$$\overline{\{h_\xi(x) \mid x \in A_\xi\}} = \overline{\{h_\xi(x) \mid x \in B_\xi\}}$$

and let $G : 2^{\omega^2} \rightarrow 2^\omega$ be any function satisfying $G(r_\xi, r) = 1$ if $r \in A_\xi$ and $G(r_\xi, r) = 0$ if $r \in B_\xi$.

Now suppose that $h : 2^\omega \rightarrow 2^\omega$ witnesses that F is universal for the function G . It is clear that the range of h must be uncountable. Hence there is ξ such that $h_\xi \subseteq h$. Then $G(r_\xi, r) = F(h(r_\xi), h_\xi(r))$ for all $r \in A_\xi \cup B_\xi$.

If f is the function defined by $f(y) = F(h(r_\xi), y)$ then f must be Baire 1 and, in particular, defining

$$C = \overline{\{h_\xi(r) \mid r \in A_\xi\}} = \overline{\{h_\xi(r) \mid r \in B_\xi\}}$$

it follows that $f \upharpoonright C$ is Baire 1. However,

$$f(h_\xi(r)) = F(h(r_\xi), h_\xi(r)) = G(r_\xi, r) = 1 \text{ for } r \in A_\xi.$$

Similarly $f(h_\xi(r)) = 0$ for $r \in B_\xi$. This is impossible for any Baire class 1 function on the perfect set C .

QED

The techniques of Miller [11] can be used to produce models with an analytic universal function but no Borel universal function.

3 Universal Functions and Martin's Axiom

Martin's Axiom implies that there are universal functions on the reals of Baire class 2, see Proposition 6.15. Here we show that weakening of Martin's axiom is not strong enough.

Lemma 3.1 *If there are models of set theory $\{\mathfrak{M}_a\}_{a \in (2^\omega)^3}$ such that:*

1. $a \in \mathfrak{M}_a$ for each $a \in (2^\omega)^3$
2. $2^\omega \not\subseteq \mathfrak{M}_a$ for each $a \in (2^\omega)^3$
3. for any $h : 2^\omega \rightarrow 2^\omega$ and any $x \in 2^\omega$ there are reals y and z such that $\{h(y), h(z)\} \subseteq \mathfrak{M}_{(x,y,z)}$

then there is no Borel universal function. Moreover, the models \mathfrak{M}_a need only be models of a sufficiently large fragment of set theory to code Borel sets by reals.

Proof

Suppose that F is a Borel universal function. Let x be a real coding it. Define $G(y, z)$ to be any element of $2^\omega \setminus \mathfrak{M}_{(x,y,z)}$. Then if $h : 2^\omega \rightarrow 2^\omega$ it is possible to find reals y and z such that $\{h(y), h(z)\} \subseteq \mathfrak{M}_{(x,y,z)}$. But then, since $\mathfrak{M}_{(x,y,z)}$ is a model of set theory, it follows that $F \in \mathfrak{M}_{(x,y,z)}$ and hence $F(h(y), h(z)) \in \mathfrak{M}_{(x,y,z)}$. Since $G(y, z) \notin \mathfrak{M}_{(x,y,z)}$ it follows that $F(h(y), h(z)) \neq G(y, z)$ and hence F can not be universal.

Theorem 3.2 *If there is a model of set theory then there is a model of set theory in which there is no Borel universal function. Indeed, there is no Borel universal function in any model obtained by forcing with a finite support product of κ^+ ccc partial orders if κ has uncountable cofinality.*

Proof

Let \mathbb{P}_α be a ccc partial order for each $\alpha \in \kappa^+$ and suppose that

$$G \subseteq \prod_{\alpha \in \kappa^+} \mathbb{P}_\alpha$$

is generic over V . Since the finite support iteration adds reals, by taking products of countably many \mathbb{P}_α it may as well be assumed that each \mathbb{P}_α adds

a real. For any $\Gamma \subseteq \kappa^+$ let V_Γ denote the model $V[G \cap \prod_{\alpha \in \Gamma} \mathbb{P}_\alpha]$. For any $x \in 2^\omega$ in $V[G]$ let $\mu(x)$ be the least ordinal such that $x \in V_{\mu(x)}$.

Given $(x, y, z) \in (2^\omega)^3$ suppose first that there is no $\theta \in \kappa$ such that $\mu(y) = \mu(x) + \theta$. In this case define $\mathfrak{M}_{(x,y,z)} = V_\xi$ where ξ is the largest of $\mu(x)$, $\mu(y)$ and $\mu(z)$. The ccc guarantees that $\xi < \kappa^+$ and the new reals added ensure that (1) and (2) of Lemma 3.1 hold. Otherwise, let $\theta(x, y) \in \kappa$ be such that $\mu(y) = \mu(x) + \theta(x, y)$. Let $\Gamma_{(x,y,z)} = \mu(z) + \theta(x, y) \cup (\kappa^+ \setminus (\mu(z) + \kappa))$ and let $\mathfrak{M}_{(x,y,z)} = V_{\Gamma_{(x,y,z)}}$. It is again clear that (1) and (2) of Lemma 3.1 hold.

To see that (3) holds suppose that $h : 2^\omega \rightarrow 2^\omega$ and $x \in 2^\omega$ are in $V[G]$. For each $\eta \in \kappa$ let $y_\eta \in 2^\omega$ be such that $\mu(y_\eta) = \mu(x) + \eta$. In other words, $\theta(x, y_\eta) = \eta$. Using the ccc, find β so large that $h(y_\eta) \in V_\beta$ for each $\eta \in \kappa$. Now let $z \in 2^\omega$ be such that $\mu(z) = \beta$ and find $\eta \in \kappa$ large enough that $h(z) \in V_\Gamma$ where $\Gamma = \beta + \eta \cup (\kappa^+ \setminus (\beta + \kappa))$. It follows that $\mathfrak{M}_{x,y_\eta,z} = V_\Gamma$ and hence $\{h(y_\eta), h(z)\} \subseteq \mathfrak{M}_{x,y_\eta,z}$. Hence (3) of Lemma 3.1 is also satisfied and the result now follows from Lemma 3.1.

Theorem 3.3 *If there is a model of set theory then there is a model of set theory in which there is no Borel universal function yet MA_{\aleph_1} holds.*

Proof

Obtain the model of MA_{\aleph_1} by iterating to ω_3 with ccc partial orders of size \aleph_1 over a model of the Continuum Hypothesis. To be precise, let $\{\mathbb{P}_\alpha\}_{\alpha \in \omega_3}$ be names for the ccc partial orders such that \mathbb{Q}_α is the iteration of $\{\mathbb{P}_\xi\}_{\xi \in \alpha}$ and \mathbb{P}_α is a \mathbb{Q}_α name for a ccc partial order of cardinality \aleph_1 . A set $\Gamma \subseteq \omega_3$ will be called *full* if for each $\gamma \in \Gamma$ all the conditions in the name \mathbb{P}_γ have support contained in $\Gamma \cap \gamma$. If Γ is full, let \mathbb{Q}_Γ be the iteration of only the partial orders \mathbb{P}_γ for $\gamma \in \Gamma$.

Cardinal arithmetic and the ccc guarantee that the partial order \mathbb{Q}_{ω_3} has the property that for any subset of \mathbb{Q}_{ω_3} of cardinality \aleph_1 is contained in completely embedded partial order of the form \mathbb{Q}_Γ where Γ is a full set of cardinality \aleph_1 . Even more, for any $\xi \in \omega_3$ if $W \subseteq \mathbb{Q}_{\omega_3}$ is such that $W \setminus \mathbb{Q}_\xi$ has cardinality \aleph_1 it is possible to find a full Γ such that $\Gamma \setminus \mathbb{Q}_\xi$ has cardinality \aleph_1 and \mathbb{Q}_Γ is completely embedded in \mathbb{Q}_{ω_3} . Using this, it is possible to mimic the proof of Theorem 3.2.

Let $G \subseteq \mathbb{Q}_{\omega_3}$ be generic and for any full $\Gamma \subseteq \omega_3$ such that \mathbb{Q}_Γ is completely embedded in \mathbb{Q}_{ω_3} let $V_\Gamma = V[G \cap \mathbb{Q}_\Gamma]$. For any $x \in 2^\omega$ in $V[G]$ let $\mu(x)$ be the least ordinal such that $x \in V_{\mu(x)}$.

Given $(x, y, z) \in (2^\omega)^3$ suppose first that there is no $\theta \in \omega_2$ such that $\mu(y) = \mu(x) + \theta$. In this case define $\mathfrak{M}_{(x,y,z)} = V_\xi$ where ξ is the largest of $\mu(x)$, $\mu(y)$ and $\mu(z)$. Otherwise, let $\theta(x, y) \in \omega_2$ be such that $\mu(y) = \mu(x) + \theta(x, y)$. There is some $\Gamma_{(x,y,z)} \subseteq \omega_3$ such that

1. $\mu(z) + \theta(x, y) \subseteq \Gamma_{(x,y,z)}$
2. $|\Gamma_{(x,y,z)} \setminus \mu(z)| = \aleph_1$
3. $\Gamma_{(x,y,z)}$ is full
4. $\mathbb{Q}_{\Gamma_{(x,y,z)}}$ is completely embedded in \mathbb{Q}_{ω_3} .

Let \mathcal{G} be the family of all $\Gamma \subseteq \omega_3$ such that

1. $\Gamma \cap \mu(z) + \omega_2 = \Gamma_{(x,y,z)} \cap \omega_2$
2. Γ is full
3. \mathbb{Q}_Γ is completely embedded in \mathbb{Q}_{ω_3} .

Let $\mathfrak{M}_{(x,y,z)} = \bigcup_{G \in \mathcal{G}} V_\Gamma$ and note that it is a model of sufficiently much set theory to code Borel sets by reals. It is again clear that (1) and (2) of Lemma 3.1 hold.

To see that (3) holds suppose that $h : 2^\omega \rightarrow 2^\omega$ and $x \in 2^\omega$ are in $V[G]$. For each $\eta \in \omega_2$ let $y_\eta \in 2^\omega$ be such that $\mu(y_\eta) = \mu(x) + \eta$. Using the ccc, find β so large that $h(y_\eta) \in V_\beta$ for each $\eta \in \omega_2$. Now let $z \in 2^\omega$ be such that $\mu(z) = \beta$ and find $\eta \in \omega_2$ large enough that $h(z) \in V_{\Gamma_{(x,y_\eta,z)}}$. It follows that $\mathfrak{M}_{x,y_\eta,z} \supseteq V_{\Gamma_{(x,y_\eta,z)}}$ and hence $\{h(y_\eta), h(z)\} \subseteq \mathfrak{M}_{x,y_\eta,z}$. Hence (3) of Lemma 3.1 is also satisfied and the result now follows from Lemma 3.1.

4 Universal Functions of special kinds

Elementary functions in the calculus of two variables can be obtained from addition $x + y$, the elementary functions of one variable and closing under composition. For example, $xy = \frac{1}{2}((x + y)^2 - x^2 - y^2)$. We might ask if there could be a universal function of the form: $F(x, y) = k(x + y)$. By this we mean that for any $G(x, y)$ we can find $u(x)$ and $v(y)$ such that $G(x, y) = k(u(x) + v(y))$.

Proposition 4.1 *If there is a universal function, then there is one of the form $F(x, y) = k(x + y)$, where k has the same complexity as the given universal function.*

Proof

For simplicity assume that $x + y$ refers to the pointwise addition in 2^ω . A similar argument can be given for ordinary addition on the real line.

Suppose $F^* : 2^\omega \times 2^\omega \rightarrow 2^\omega$ is a universal function, i.e., for every $f : 2^\omega \times 2^\omega \rightarrow 2^\omega$ there are g, h with $f(x, y) = F^*(g(x), h(y))$ all $x, y \in 2^\omega$. Given any $u \in 2^\omega$ let u_0 be u shifted onto the even coordinates, i.e., $u_0(2n) = u(n)$ and $u_0(2n + 1) = 0$. Similarly for $v \in 2^\omega$ let v_1 be v shifted onto the odd coordinates. Note that (u, v) is easily recovered from $u_0 + v_1$. Hence we can define k by $k(w) = F^*(u, v)$ where $w = u_0 + v_1$.

QED

Proposition 4.2 *Suppose that there is a universal function $F : 2^\omega \times 2^\omega \rightarrow 2^\omega$. Then there exists a function $f : 2^\omega \rightarrow 2^\omega$ such that for every symmetric $H : 2^\omega \times 2^\omega \rightarrow 2^\omega$ there exists a $g : 2^\omega \rightarrow 2^\omega$ such that $H(x, y) = f(g(x) + g(y))$ for every two distinct $x, y \in 2^\omega$. Furthermore if F is Borel, then f is Borel.*

Proof

Let $P_s \subseteq \omega$ for $s \in 2^{<\omega}$ partition ω into infinite sets. We say that $y : P_s \rightarrow 2$ codes $x : \omega \rightarrow 2$ iff $y(a_n) = x(n)$ where $a_0 < a_1 < a_2 < \dots$ is the increasing listing of P_s .

For any $x \in 2^\omega$ define $q(x) \in 2^\omega$ so that $q(x) \upharpoonright P_{x \upharpoonright n}$ codes x for every $n < \omega$ and $q(x) \upharpoonright P_s$ is identically 0 for any s which is not an initial segment of x .

By assumption for any $H : 2^\omega \times 2^\omega \rightarrow 2^\omega$ there exists h such that

$$H(x, y) = F(h(x), h(y))$$

for all $x, y \in 2^\omega$. Define $g(x) = q(h(x))$. Without loss of generality we may assume that h is one-to-one and never identically 0. Notice for $x \neq y$ that we may easily recover $h(x)$ and $h(y)$ from $q(x) + q(y)$. (There will be exactly two infinite paths in the set of all $s \in 2^{<\omega}$ such that $(q(x) + q(y)) \upharpoonright P_s$ is not identically 0). Hence, we may define f so that

$$f(g(x) + g(y)) = F(h(x), h(y)).$$

QED

Proposition 4.3 *There does not exist a Borel function $F : 2^\omega \times 2^\omega \rightarrow 2^\omega$ such that for every Borel $H : 2^\omega \times 2^\omega \rightarrow 2^\omega$ there exists Borel $g, h : 2^\omega \rightarrow 2^\omega$ with*

$$H(x, y) = F(g(x), h(y))$$

for every $x, y \in 2^\omega$.

Proof

Suppose F is a Baire level α function and let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\Sigma_{\alpha+3}^0$ set. Let H be the characteristic function of U . Given any Borel g, h let $P \subseteq 2^\omega$ be perfect set on which h is continuous. Fix x_0 so that $U_{x_0} \subseteq P$ is not $\Delta_{\alpha+3}^0$. But if we define

$$q : P \rightarrow 2^\omega \text{ by } q(y) = F(g(x_0), h(y))$$

then q is Baire level α and $U_{x_0} = q^{-1}(1)$ which is a contradiction.

QED

This proof is similar to Mansfield and Rao's Theorem [7, 8, 15] that the universal analytic set in the plane is not in the σ -algebra generated by rectangles with measurable sides. See also, Miller [12].

Question 4.4 *Does there always exists a Borel function*

$$F : 2^\omega \times 2^\omega \rightarrow 2^\omega$$

such that for every Borel $H : 2^\omega \times 2^\omega \rightarrow 2^\omega$ there exists $g, h : 2^\omega \rightarrow 2^\omega$ with

$$H(x, y) = F(g(x), h(y))$$

for every $x, y \in 2^\omega$?

Maybe Louveau's Theorem [6] is relevant for this question.

Stevo Todorcevic has noted the following version of universal functions:

There exists continuous functions $F_n : 2^\omega \times 2^\omega \rightarrow 2^\omega$ for $n < \omega$ with the property that for every $G : \mathfrak{p} \times \mathfrak{p} \rightarrow 2^\omega$ there exists $h : \mathfrak{p} \rightarrow 2^\omega$ such that for every $\alpha, \beta < \mathfrak{p}$

$$G(\alpha, \beta) = \lim_{n \rightarrow \infty} F_n(h(\alpha), h(\beta))$$

Where \mathfrak{p} is the least cardinal for which MA σ -centered fails.

We prove that this sort of universal function is equivalent to a level 2 universal function.

Proposition 4.5 *For any cardinal κ the following are equivalent:*

(1) *There exists continuous functions $F_n : 2^\omega \times 2^\omega \rightarrow 2^\omega$ for $n < \omega$ with the property that for every $G : \kappa \times \kappa \rightarrow 2^\omega$ there exists $h : \kappa \rightarrow 2^\omega$ such that*

$$G(\alpha, \beta) = \lim_{n \rightarrow \infty} F_n(h(\alpha), h(\beta)) \text{ for every } \alpha, \beta \in \kappa.$$

(2) *There exists a level-2 function $F : 2^\omega \times 2^\omega \rightarrow 2^\omega$ with the property that for every $G : \kappa \times \kappa \rightarrow 2^\omega$ there exists $h : \kappa \rightarrow 2^\omega$ such that*

$$G(\alpha, \beta) = F(h(\alpha), h(\beta)) \text{ for every } \alpha, \beta \in \kappa.$$

Proof

Recall that F is level 2 means that for every n the set

$$\{(x, y) : F(x, y)(n) = 1\} \text{ is } F_\sigma.$$

(1) \rightarrow (2). Given the sequence F_k of continuous functions define:

$$F(x, y)(n) = 1 \text{ iff } F_k(x, y)(n) = 1 \text{ for all but finitely many } k < \omega.$$

(2) \rightarrow (1). For any G let G_0 be G and define G_1 by $G_1(\alpha, \beta)(n) = 1 - G_0(\alpha, \beta)(n)$. That is, we switch 0 and 1 on every coordinate of the output. It follows that we have h_0 and h_1 such that for ever $\alpha, \beta < \kappa$ and $n < \omega$

$G(\alpha, \beta)(n) = 1$ implies

$$F(h_0(\alpha), h_0(\beta))(n) = 1 \text{ and } F(h_1(\alpha), h_1(\beta))(n) = 0$$

$G(\alpha, \beta)(n) = 0$ implies

$$F(h_0(\alpha), h_0(\beta))(n) = 0 \text{ and } F(h_1(\alpha), h_1(\beta))(n) = 1$$

For each n define the pair of (nondisjoint) F_σ sets P_n^0 and P_n^1 by

$$(\langle u_0, u_1 \rangle, \langle v_0, v_1 \rangle) \in P_n^i \text{ iff } F(u_i, v_i)(n) = 1$$

By the reduction property for each n there are disjoint F_σ sets Q_n^0, Q_n^1 with $Q_n^i \subseteq P_n^i$ and $Q_n^0 \cup Q_n^1 = P_n^0 \cup P_n^1$. Write each Q_n^i as an increasing sequence of closed sets $Q_n^i = \bigcup_k C_{n,k}^i$. Since $C_{n,k}^0$ and $C_{n,k}^1$ are disjoint closed sets, there is a clopen set $D_{n,k}$ with $C_{n,k}^0 \subseteq D_{n,k}$ and $C_{n,k}^1$ disjoint from $D_{n,k}$.

Define the continuous map F_k as follows:

$$F_k(u, v)(n) = 1 \text{ iff } (u, v) \in D_{n,k}$$

Now we verify that this works. Given G take h_0 and h_1 as above and define $h(\gamma) = \langle h_0(\gamma), h_1(\gamma) \rangle$. Then for any α, β, n

If $G(\alpha, \beta)(n) = 1$, then $\langle h(\alpha), h(\beta) \rangle \in P_n^0 \setminus P_n^1$ so $\langle h(\alpha), h(\beta) \rangle \in Q_n^0$ and so $F_k(\langle h(\alpha), h(\beta) \rangle)(n) = 1$ for all but finitely many k .

If $G(\alpha, \beta)(n) = 0$, then $\langle h(\alpha), h(\beta) \rangle \in P_n^1 \setminus P_n^0$ so $\langle h(\alpha), h(\beta) \rangle \in Q_n^1$ and so $F_k(\langle h(\alpha), h(\beta) \rangle)(n) = 0$ for all but finitely many k .

QED

Davies [3] showed that the continuum hypothesis implies that the function

$$F(\vec{x}, \vec{y}) = \sum_{n < \omega} x_n y_n$$

has a universal property: for every $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ there are functions f_n, g_n for $n < \omega$ such that

$$H(x, y) = \sum_{n < \omega} f_n(x) g_n(y)$$

for all $x, y \in \mathbb{R}$. Moreover the sum has only finitely many nonzero terms. Shelah [16] remarks that Davies result is false in the Cohen real model.

5 Abstract Universal Functions

Theorem 5.1 *If κ is an infinite cardinal such that $2^{<\kappa} = \kappa$, then there is a universal function $F : \kappa \times \kappa \rightarrow \kappa$.*

Proof

Choose $\rho_\alpha : \kappa \times \kappa \rightarrow \kappa$ for $\alpha < \kappa$ with the property that for every $\beta < \kappa$ and $k : \beta \rightarrow \kappa$ there exists a $\alpha < \kappa$ with $k = \rho_\alpha \upharpoonright \beta$. Let $\langle \cdot, \cdot \rangle : \kappa \times \kappa \rightarrow \kappa$ be a bijective pairing function. Define $F : \kappa \times \kappa \rightarrow \kappa$ as follows:

$$F(\langle \alpha_0, \alpha_1 \rangle, \langle \beta_0, \beta_1 \rangle) = \begin{cases} \rho_{\beta_1}(\alpha_0) & \text{if } \alpha_0 \leq \beta_0 \\ \rho_{\alpha_1}(\beta_0) & \text{if } \alpha_0 > \beta_0 \end{cases}$$

To see that F is universal, let $H : \kappa \times \kappa \rightarrow \kappa$ be arbitrary. For each β choose $h(\beta)$ so that $H(\alpha, \beta) = \rho_{h(\beta)}(\alpha)$ for all $\alpha \leq \beta$. Similarly, choose $g(\alpha)$ so that $H(\alpha, \beta) = \rho_{g(\alpha)}(\beta)$ for all $\beta < \alpha$. It follows that

$$H(\alpha, \beta) = F(\langle \alpha, g(\alpha) \rangle, \langle \beta, h(\beta) \rangle)$$

for all $\alpha, \beta < \kappa$.

QED

Remark 5.2 *For example, there is a universal $F : \omega \times \omega \rightarrow \omega$.*

Remark 5.3 *Theorem 5.1 is probably just a special case of Theorem 6 of Rado [13].*

Definition 5.4 *Let $Fin(X)$ be the partial order of partial functions from a finite subset of X into 2. Let $Ctbl(Y)$ be the partial order³ of countable partial functions from Y into 2.*

Theorem 5.5 *It is relatively consistent with ZFC that there is no universal function $F : \mathfrak{c} \times \mathfrak{c} \rightarrow \mathfrak{c}$.*

Proof

In our model $\mathfrak{c} = \omega_2$ and there is no $F : \omega_2 \times \omega_2 \rightarrow 2$ with the property that for every $f : \omega_2 \times \omega_1 \rightarrow 2$ there exists $g_1 : \omega_2 \rightarrow \omega_2$ and $g_2 : \omega_1 \rightarrow \omega_2$ such that $f(\alpha, \beta) = F(g_1(\alpha), g_2(\beta))$ for every $\alpha < \omega_2$ and $\beta < \omega_1$.

Let M be a countable transitive model of ZFC + GCH. Force with $Ctbl(\omega_3)$ followed by $Fin(\omega_2)$. Let G be $Ctbl(\omega_3)$ -generic over M and H be $Fin(\omega_2)$ -generic over $M[G]$. We will show there is no F in the model $N = M[G][H]$.

By standard arguments⁴ involving iteration and product forcing we may regard N as being obtained by forcing with $Ctbl(\omega_3)^M$ over the ground model $M[H]$. Of course, in $M[H]$ the poset $Ctbl(\omega_3)^M$ is not countably closed but it still must have the ω_2 -cc. Hence for any $F : \omega_2 \times \omega_2 \rightarrow 2$ in N we may find $\gamma < \omega_3$ such that $F \in M[H][G \restriction \gamma]$.

Use G above γ to define $f : \omega_2 \times \omega_1 \rightarrow 2$, i.e.,

$$f(\alpha, \beta) = G(\gamma + \omega_1 \cdot \alpha + \beta).$$

Now suppose for contradiction that in N there were $g_1 : \omega_2 \rightarrow \omega_2$ and $g_2 : \omega_1 \rightarrow \omega_2$ such that $f(\alpha, \beta) = F(g_1(\alpha), g_2(\beta))$ for every $\alpha < \omega_2$ and $\beta < \omega_1$. Using the ω_2 chain condition there would be $I \subseteq \omega_3$ in M of size ω_1 such that $g_2 \in M[H][G \restriction (\gamma \cup I)]$. Choose $\alpha_0 < \omega_2$ so that $\gamma \cup I$ is disjoint from

$$D = \{\gamma + \omega_1 \cdot \alpha_0 + \beta : \beta < \omega_1\}.$$

It is easy to see by a density argument that the function $G \restriction D$ is not in $M[H][G \restriction (\gamma \cup I)]$. But this is a contradiction, since $G \restriction D$ is easily defined

³These posets are denoted in Kunen[5] by $Fn(X, 2)$ and $Fn(Y, 2, \omega_1)$.

⁴Kunen[5] p.253 Solovay [21] p.10

from the function $f(\alpha_0, \cdot)$, $f(\alpha_0, \beta) = F(g_1(\alpha_0), g_2(\beta))$ for all β , and F, g_2 are in $M[H][G \restriction (\gamma \cup I)]$.

QED

Question 5.6 *Is it consistent with $2^{<\mathfrak{c}} > \mathfrak{c}$ to have a universal function $F : 2^\omega \times 2^\omega \rightarrow 2^\omega$? How about a Borel F ?*

Theorem 5.7 *Suppose MA_{ω_1} . Then there exists $F : \omega_1 \times \omega \rightarrow \omega_1$ which is universal, i.e., for every $f : \omega_1 \times \omega \rightarrow \omega_1$ there exists $g : \omega_1 \rightarrow \omega_1$ and $h : \omega \rightarrow \omega$ such that*

$$f(\alpha, n) = F(g(\alpha), h(n)) \text{ for every } \alpha < \omega_1 \text{ and } n < \omega.$$

Proof

There is an obvious notion of universal $F : \alpha \times \beta \rightarrow \gamma$. We produce a universal $F : \omega_1 \times \omega \rightarrow \omega$ and then show that this is equivalent to the existence of a universal $F : \omega_1 \times \omega \rightarrow \omega_1$.

Standard arguments, show that there exists a family $h_\alpha : \omega \rightarrow \omega$ for $\alpha < \omega_1$ of independent functions, i.e., for any n , $\alpha_1 < \alpha_2 < \dots < \alpha_n < \omega_1$ and $s : \{1, \dots, n\} \rightarrow \omega$ there are infinitely many $k < \omega$ such that

$$h_{\alpha_1}(k) = s(1)$$

$$h_{\alpha_2}(k) = s(2)$$

$$\vdots$$

$$h_{\alpha_n}(k) = s(n).$$

Define $H : \omega_1 \times \omega \rightarrow \omega$ by $H(\alpha, n) = h_\alpha(n)$. We show that H is universal mod finite, in sense which will be made clear. Given any $f : \omega_1 \times \omega \rightarrow \omega$ define the following poset \mathbb{P} . A condition $p = (s, F)$ is a pair such that $s \in \omega^\omega$ is one-to-one and $F \in [\omega_1]^{<\omega}$. We define $p \leq q$ iff

1. $s_q \subseteq s_p$,
2. $F_q \subseteq F_p$, and
3. $f(\alpha, n) = h_\alpha(s_p(n))$ for every $\alpha \in F_q$ and $n \in \text{dom}(s_p) \setminus \text{dom}(s_q)$.

It is easy to see that \mathbb{P} is ccc, in fact, σ -centered since any two conditions with the same s are compatible. Since the family $(h_\alpha : \omega \rightarrow \omega : \alpha < \omega_1)$ is independent, for any $p \in \mathbb{P}$ there are extensions of p with arbitrarily long s part. It follows from MA_{ω_1} that there exists $h : \omega \rightarrow \omega$ with the property that for every $\alpha < \omega_1$ for all but finitely many n that $f(\alpha, n) = h_\alpha(h(n))$.

To get a universal map $F : \omega_1 \times \omega \rightarrow \omega$, simply take any F with the property that for every $\alpha < \omega_1$ and any $h' =^* h_\alpha$ (equal mod finite) there is β such that $F(\beta, n) = h'(n)$ for every n . Since the function h is one-to-one, it easy to find $k : \omega_1 \rightarrow \omega_1$ such that $F(k(\alpha), h(n)) = f(\alpha, n)$ for all α and n .

Finally we show that having a universal $F : \omega_1 \times \omega \rightarrow \omega$ gives a universal $F' : \omega_1 \times \omega \rightarrow \omega_1$. For any infinite $\alpha < \omega_1$ fix a bijection from $j_\alpha : \omega \rightarrow \alpha$. Construct F' with the property that for every pair $\alpha, \beta < \omega_1$ there are uncountably many $\gamma < \omega_1$ such that $F'_\gamma = j_\beta \circ F_\alpha$, i.e.,

$$F'(\gamma, n) = j_\beta(F(\alpha, n)) \text{ for all } n < \omega.$$

Now we verify that F' is universal. Let $f' : \omega_1 \times \omega \rightarrow \omega_1$ be arbitrary. Let

$$i_\alpha = \omega + \sup\{f'(\alpha, n) + 1 : n < \omega\}.$$

Define f into ω by $f(\alpha, n) = j_{i_\alpha}^{-1}(f'(\alpha, n))$. Since F is universal there exists g, h with

$$F(g(\alpha), h(n)) = f(\alpha, n) = j_{i_\alpha}^{-1}(f'(\alpha, n)).$$

By our definition of F' we may construct g' so that

$$F'(g'(\alpha), h(n)) = j_{i_\alpha}(F(g(\alpha), h(n)))$$

and we are done since

$$j_{i_\alpha}(F(g(\alpha), h(n))) = f'(\alpha, n).$$

QED

Question 5.8 Does MA_{ω_1} imply there exists $F : \omega_1 \times \omega_1 \rightarrow \omega_1$ which is universal? Is it consistent one way or the other? This question may be related to Shelah results on universal graphs of size ω_1 , see Shelah [17, 18, 19].

Proposition 5.9 If there is a universal function $F : \kappa \times \gamma \rightarrow 2$ then for every $n < \omega$ there is a universal function $F : \kappa \times \gamma \rightarrow n$.

Proof

We produce a F^* which is universal for $n = 2 \times 2$. For any $H_1, H_2 : \kappa \times \gamma \rightarrow 2$ there exists g_1, h_1, g_2, h_2 such that

$$H_1(\alpha, \beta) = F(g_1(\alpha), h_1(\beta)) \text{ and } H_2(\alpha, \beta) = F(g_2(\alpha), h_2(\beta))$$

for all $\alpha \in \kappa$ and $\beta \in \gamma$. Now define

$$F^*(\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle) = \langle F(\alpha_1, \beta_1), F(\alpha_2, \beta_2) \rangle$$

and

$$g(\alpha) = \langle g_1(\alpha), g_2(\alpha) \rangle \text{ and } h(\beta) = \langle h_1(\beta), h_2(\beta) \rangle.$$

Note that

$$F^*(g(\alpha), h(\beta)) = \langle H_1(\alpha, \beta), H_2(\alpha, \beta) \rangle$$

for all $\alpha \in \kappa$ and $\beta \in \gamma$.

QED

6 Higher Dimensional Universal Functions

Definition 6.1 A k -dimensional universal function is a function

$$F : (2^\omega)^k \rightarrow 2^\omega$$

such that for every function $G : (2^\omega)^k \rightarrow 2^\omega$ there is $h : 2^\omega \rightarrow 2^\omega$ such that

$$G(x_1, x_2, \dots, x_k) = F(h(x_1), h(x_2), \dots, h(x_k))$$

for all $(x_1, x_2, \dots, x_k) \in (2^\omega)^k$.

Proposition 6.2 Suppose $F(x, y)$ is a universal function, then $F(F(x, y), z)$ is a 3-dimensional universal function.

Proof

Given $G(x, y, z)$ define $G_0(u, z) = G(u_0, u_1, z)$ using unpairing, $u = \langle u_0, u_1 \rangle$. By universality of F there are g, h with $G_0(u, z) = F(g(u), h(z))$. Again by universality of F there are g_0, g_1 with $g(\langle u_0, u_1 \rangle) = F(g_0(u_0), g_1(u_1))$ and hence $G(x, y, z) = F(F(g_0(x), g_1(y)), h(z))$.

QED

Hence the existence of a universal function in dimension 2 is equivalent to the existence of a universal function in dimension k for any $k > 1$. Note however that the Baire complexity of $F(F(x, y), z)$ is higher than that of F .

We may also consider universal functions F where the parameters functions are functions of more than one variable, for example:

$$\forall G \exists g, h, k \forall x, y, z \quad G(x, y, z) = F(g(x, y), h(y, z), k(z, x)).$$

Although this easily follows from the existence of a dimension 3 universal, we do not know if it is equivalent. The reader will easily be able to imagine many variants. For example,

$$G(x, y, z) = F(g(x, y), h(y, z))$$

$$G(x_1, x_2, x_3, x_4) = F(g_1(x_1, x_2), g_2(x_2, x_3), g_3(x_3, x_4), g_4(x_4, x_1))$$

where we have omitted quantifiers for clarity. These two variants are equivalent to the existence of 2-dimensional universal function. To see this in the first example put $y = 0$ and get

$$G(x, z) = F(g(x, 0), h(0, z)).$$

In the second example put $x_2 = x_4 = 0$ and get

$$G(x_1, x_3) = F(g_1(x_1, 0), g_2(0, x_3), g_3(x_3, 0), g_4(0, x_1)).$$

More generally, suppose F and \vec{x}_k 's have the property that for every G there are g_k 's such that for all \vec{x}

$$G(\vec{x}) = F(g_1(\vec{x}_1), \dots, g_n(\vec{x}_n)).$$

Suppose there are two variables x and y from \vec{x} which do not simultaneously belong to any \vec{x}_k . Then we get a universal 2-dimensional function simply by putting all of the other variables equal to zero.

Proposition 6.3 *If there is a $(3, 2)$ -dimensional universal function, i.e., an $F(x, y, z)$ such that for every G there is h with*

$$G(x, y, z) = F(h(x, y), h(y, z), h(z, x)) \text{ all } x, y, z$$

then for every $n > 3$ there is a $(n, 2)$ -dimensional universal function F , i.e., for every G n -ary there is a binary h with

$$G(x_1, x_2, \dots, x_n) = F(\langle h(x_i, x_j) : 1 \leq i < j \leq n \rangle) \text{ all } \vec{x}.$$

F is $\binom{n}{2}$ -ary. Conversely, if there is a $(n, 2)$ -dimensional universal function for some $n > 3$, then there is a $(3, 2)$ -dimensional universal function.

Proof

Consider the case for $n = 4$.

Suppose that F is $(3, 2)$ -dimensional universal function. Given a 4-ary function $G(x, y, z, w)$ for each fixed w we get a function $h_w(u, v)$ with

$$G(x, y, z, w) = F(h_w(x, y), h_w(y, z), h_w(z, x)) \text{ for all } x, y, z.$$

But now considering $h(u, v, w) = h_w(u, v)$ we get a function $k(s, t)$ with $h(u, v, w) = F(k(u, v), k(v, w), k(w, u))$. Note that

$$\begin{aligned} G(x, y, z, w) = & F(F(k(x, y), k(y, w), k(w, x)), \\ & F(k(y, z), k(z, w), k(w, y)), \\ & F(k(z, x), k(x, w), k(w, z))). \end{aligned}$$

Note that $k(s, t)$ and $k(t, s)$ can be combined by pairing and unpairing into a single function $k_1(s, t)$. From this one can define a $(4, 2)$ -dimensional universal function.

For the converse, if F is a $(4, 2)$ -dimensional universal function, then for every G 3-ary, there exists h binary with

$$G(x, y, z) = F(h(x, y), h(y, z), h(x, z), h(x, 0), h(y, 0), h(z, 0)).$$

But note that, for example, $h(x, y)$ and $h(x, 0)$ can be combined into a single function of $h_1(x, y)$. Hence we can get a $(3, 2)$ -dimensional universal function. QED

Next we state a generalization of these ideas:

Definition 6.4 Suppose $\Sigma \subseteq \mathcal{P}(\{0, 1, 2, \dots, n-1\}) = \mathcal{P}(n)$ (the power set of n). Define $U(\kappa, n, \Sigma)$ to mean that there exists $F : \kappa^\Sigma \rightarrow \kappa$ such that for every $G : \kappa^n \rightarrow \kappa$ there are $h_Q : \kappa^{|Q|} \rightarrow \kappa$ for $Q \in \Sigma$ such that

$$G(x_0, x_1, \dots, x_{n-1}) = F(h_Q(x_j : j \in Q) : Q \in \Sigma) \text{ for all } \vec{x} \in \kappa^n.$$

Then the last two propositions can be generalized to show:

Proposition 6.5 For any infinite cardinal κ and positive integer n

1. $U(\kappa, n+1, [n+1]^n)$ implies $\forall m > n \ U(\kappa, m, [m]^n)$.
2. $(\exists m > n \ U(\kappa, m, [m]^n))$ implies $U(\kappa, n+1, [n+1]^n)$.

3. $U(\kappa, n+1, [n+1]^n)$ implies $U(\kappa, n+2, [n+2]^{n+1})$

Definition 6.6 Define $U(\kappa, n)$ to be any of the equivalent $U(\kappa, m, [m]^n)$ for $m > n$. Note that n is the arity of the inside parameter functions, the arity of the universal function is less important.

We will show that $U(\kappa, n)$ are the only generalized multi-dimensional universal functions properties. Clause (3) says that $U(\kappa, n)$ implies $U(\kappa, n+1)$ and we will show that none of these implications can be reversed.

Proposition 6.7 Let κ be an infinite cardinal, $n \geq 2$, and $\Sigma, \Sigma_0, \Sigma_1$ subsets of $\mathcal{P}(n)$.

1. If $\Sigma_0 \subseteq \Sigma_1$, then $U(\kappa, n, \Sigma_0)$ implies $U(\kappa, n, \Sigma_1)$.
2. If $Q_0 \subseteq Q_1 \in \Sigma$, then $U(\kappa, n, \Sigma)$ is equivalent to $U(\kappa, n, \Sigma \cup \{Q_0\})$.
3. Suppose Σ is closed under taking subsets, every element of n is in some element of Σ , and $n = \{0, 1, 2, \dots, n-1\} \notin \Sigma$. Let $m+1$ be the size of the smallest subset of n not in Σ . Then $U(\kappa, n, \Sigma)$ is equivalent to $U(\kappa, m)$.

Proof

(1) This is true because the F which works for Σ_0 also works for Σ_1 by ignoring the values of h_Q for $Q \in \Sigma_1 \setminus \Sigma_0$.

(2) This is true because given h_{Q_0}, h_{Q_1} we may define a new \hat{h}_{Q_1} by outputting the pairing

$$\hat{h}_{Q_1}(x_j : j \in Q_1) = \langle (h_{Q_0}(x_j : j \in Q_0), (h_{Q_1}(x_j : j \in Q_1)) \rangle$$

(3) Choose $R \subseteq \{0, 1, \dots, n-1\}$ not in Σ with $|R| = m+1$. By choice of m all subsets of R of size m are in Σ . By setting $x_i = 0$ for $i \notin R$, we see that $U(\kappa, m)$ is true.

Now assume $U(\kappa, m)$ is true. By Proposition 6.5 we have that $U(\kappa, n, \Sigma_0)$ is true where $\Sigma_0 = [n]^m$. But $\Sigma_0 \subseteq \Sigma$ and so by part (1), $U(\kappa, n, \Sigma)$ is true. QED

Remark 6.8 For any n and $\Sigma \subseteq \mathcal{P}(n)$ if $\bigcup \Sigma \neq n = \{0, 1, 2, \dots, n-1\}$ then $U(\kappa, n, \Sigma)$ is trivially false. If $n \in \Sigma$, then $U(\kappa, n, \Sigma)$ is trivially true. If neither of these is true, then by the Proposition 6.7 there exists m with $U(\kappa, n, \Sigma)$ equivalent to $U(\kappa, m)$.

Proposition 6.9 *The following are true in ZFC.*

1. $U(\omega, 1)$
2. $U(\omega_1, 2)$
3. $U(\kappa, 1)$ implies $U(\kappa^+, 2)$
4. $U(\kappa, n)$ implies $U(\kappa^+, n + 1)$
5. $U(\omega_n, n + 1)$ every $n \geq 0$.

Proof

For (1) see Remark 5.2. We prove (2) and leave 3-5 to the reader.

Suppose that $f : \omega^2 \rightarrow \omega$ witnesses $U(\omega, 2, 1)$. For any countable ordinal $\delta > 0$ let $\delta = \{\delta_i : i < \omega\}$. Define

$$F_0(\delta, n, m) = \delta_{f(n, m)}.$$

Now suppose $G : \omega_1^3 \rightarrow \omega_1$. Define

$$k(\delta) = \sup\{G(\alpha, \beta, \gamma) : \alpha, \beta, \gamma \leq \delta\} + 1$$

For any $\gamma < \omega_1$ let $\gamma^* = k(\gamma)$. Define $g : \omega^2 \rightarrow \omega$ by

$$G((\gamma + 1)_n, (\gamma + 1)_m, \gamma) = \gamma_{g(n, m)}^*.$$

By the universality property of f there exists $h : \omega \rightarrow \omega$ with

$$g(n, m) = f(h(n), h(m)) \text{ for every } n, m \in \omega.$$

For $\delta \leq \gamma$ define $h_1(\delta, \gamma) = h(k)$ where $\delta = (\gamma + 1)_k$. Then we have that

$$\forall \alpha, \beta \leq \gamma < \omega_1 \quad G(\alpha, \beta, \gamma) = F_0(k(\gamma), h_1(\alpha, \gamma), h_1(\beta, \gamma)).$$

Define F as follows:

$$F(\alpha, \beta, \gamma, \alpha^*, \beta^*, \gamma^*, n_1, m_1, n_2, m_2, n_3, m_3) =$$

$$\begin{cases} F_0(\gamma^*, n_1, m_1) & \text{if } \alpha, \beta \leq \gamma \\ F_0(\beta^*, n_2, m_2) & \text{if } \gamma < \beta \text{ and } \alpha \leq \beta \\ F_0(\alpha^*, n_3, m_3) & \text{if } \beta, \gamma < \alpha \end{cases}$$

Then given G we can find k, h_1, h_2, h_3 so that

$$G(\alpha, \beta, \gamma) = F(\alpha, \beta, \gamma, k(\alpha), k(\beta), k(\gamma), h_1(\alpha, \gamma), h_1(\beta, \gamma), h_2(\alpha, \beta), h_2(\gamma, \beta), h_3(\beta, \alpha), h_3(\gamma, \alpha)).$$

QED

The κ -Cohen real model is any model of ZFC obtained by forcing with the poset of finite partial functions from κ to 2 over a countable transitive ground model satisfying ZFC.

Proposition 6.10 *In the ω_2 -Cohen real model we have that $U(\omega_1, 1)$ fails. Similarly, $U(\omega_2, 2)$ fails in the ω_3 -Cohen real model. More generally, we have that $U(\gamma, n)$ fails in the κ -Cohen real model when $\kappa > \gamma \geq \omega_n$.*

Proof

We show that $U(\omega_2, 2)$ fails in the ω_3 -Cohen real model, leaving the rest to the reader.

Let M be a countable transitive model of ZFC and in M define \mathbb{P} to be the poset of finite partial maps from $\omega_3 \times \omega_3 \times \omega_3$ into 2. We claim that if G is \mathbb{P} -generic over M , then there is no map $F : \omega_2 \times \omega_2 \times \omega_2 \rightarrow \omega_2$ which is (3,2)-universal for maps of the form $H : \omega \times \omega_1 \times \omega_2 \rightarrow 2$.

Suppose for contradiction that F is such a map. By the ccc we may find $\gamma_0 < \omega_3$ with $F \in M[G \restriction \gamma_0^3]$. Hence we may find maps $h_1 : \omega \times \omega_1 \rightarrow \omega_3$, $h_2 : \omega \times \omega_2 \rightarrow \omega_3$, and $h_3 : \omega_1 \times \omega_2 \rightarrow \omega_3$ such that

$$H(n, \beta, \gamma) \stackrel{\text{def}}{=} G(n, \beta, \gamma_0 + \gamma) = F(h_1(n, \beta), h_2(n, \gamma), h_3(\beta, \gamma)).$$

for every $n < \omega, \beta < \omega_1, \gamma < \omega_2$. By ccc we can choose $\gamma_1 < \omega_2$ such that $h_1 \in M[G^*]$ where G^* is G restricted to $\{(\alpha, \beta, \rho) \in \omega^3 : \rho \neq \gamma_0 + \gamma_1\}$. Define $g : \omega \times \omega_1 \rightarrow 2$ by

$$g(n, \alpha) = G(n, \alpha, \gamma_0 + \gamma_1)$$

Note that we have that $F, h_1 \in M[G^*]$, g is Cohen generic over $M[G^*]$, and

$$g(n, \alpha) = F(h_1(n, \alpha), h_2(n, \gamma_0 + \gamma_1), h_3(\alpha, \gamma_0 + \gamma_1)).$$

Since the extension by g is ccc, we may find $\alpha_0 < \omega_1$ such that

$$h_2 \in M[G^*][g \restriction (\omega \times \alpha_0)] \stackrel{\text{def}}{=} N.$$

But this is a contradiction because g_{α_0} defined by $g_{\alpha_0}(n) = g(n, \alpha_0)$ is Cohen generic over N . But $F, h_1, h_2 \in N$ and for any $\gamma_2 < \omega_2$ the map k defined by

$$k(n) = F(h_1(n, \alpha_0), h_2(n, \gamma_0 + \gamma_1), \gamma_2) \text{ for all } n < \omega$$

is in N and so can never be equal to g_{α_0} . Thus $h_3(\alpha_0, \gamma_0 + \gamma_1) = \gamma_2$ cannot be defined.

QED

Corollary 6.11 *Let $\aleph_\omega \leq \gamma < \kappa$. In the κ -Cohen real model we have that*

$$U(\omega_n, n+1) + \neg U(\omega_n, n) \text{ for all } n > 0,$$

and

$$\neg U(\gamma, n) \text{ for all } n > 0.$$

Remark 6.12 *If we start with a model M_1 of GCH and force with the countable partial functions from $\kappa = \aleph_{\omega+1}$ into 2 then in the resulting model M_2 , we have CH and so $U(\omega_1, 1)$ (Theorem 5.1). We get $U(\omega_n, n)$ by Propositions 6.9. By an argument similar to Proposition 6.10 but raised up one cardinal, we have $\neg U(\omega_n, n-1)$ for $n \geq 2$. If we then add $\kappa = \omega_3$ Cohen reals to M_2 to get M_3 , then we will have in M_3 that $|2^\omega| = \omega_3$ and $\neg U(\omega_3, 2)$ by argument of Proposition 6.10 lifted by one cardinal. $U(\omega_3, 4)$ is true in ZFC by Proposition 6.9. This leaves the obvious gap question.*

Definition 6.13 *In the case of Borel universal functions of higher dimensions, we use $U(\text{Borel}, n)$ to mean the analogous thing as in Definition 6.6 only we require that the universal map F be Borel.*

Proposition 6.14 *The following are true:*

1. $U(\text{Borel}, n)$ implies $U(\text{Borel}, n+1)$
2. $U(\text{Borel}, \Sigma, n)$ is equivalent to $U(\text{Borel}, m)$ for $m+1$ the size of the smallest subset of n not in the downward closure of Σ .

Proof

The composition of Borel functions is Borel, and pairing and unpairing functions are continuous.

QED

We can further refine $U(\text{Borel}, n)$ in the special case that our universal function F is a level α Borel function. Since the composition of level α -functions is not necessarily level α , i.e., $F(F(x, y), z)$ need be at the level α just because F is. Hence it is not immediately obvious that the binary case of the next proposition implies the n -ary case. The proof here is similar to that of Rao [14].

Proposition 6.15 *Assume Martin's Axiom. Then for every $n > 1$ there is a level 2 Borel function $F : (2^\omega)^n \rightarrow 2^\omega$ which is universal, i.e., for every $G : (2^\omega)^n \rightarrow 2^\omega$ there exists $h_i : 2^\omega \rightarrow 2^\omega$ such that for every x in $(2^\omega)^n$*

$$G(x_1, \dots, x_n) = F(h_1(x_1), \dots, h_n(x_n))$$

Proof

For simplicity we prove it for $n = 3$. Let

$$D_1 = \{(\alpha, \beta, \gamma) : \alpha, \beta \leq \gamma < \mathfrak{c}\}$$

Let $F \subseteq 2^\omega \times 2^\omega \times 2^\omega$ be an F_σ set with the property that for every F_σ set $H \subseteq 2^\omega \times 2^\omega$ there exists $z \in 2^\omega$ with

$$H = F_z =^{def} \{(x, y) : (x, y, z) \in F\}.$$

Let $g : \mathfrak{c} \rightarrow 2^\omega$ be a 1-1 map. Recall that Martin's Axiom implies that every set $X \subseteq 2^\omega$ with $|X| < \mathfrak{c}$ is a \mathcal{Q} -set, i.e., every $Y \subseteq X$ is a relative F_σ . This is due to Silver and can be found in any standard treatment of Martin's Axiom. Thus given any $A \subseteq D_1$ we can find $h_1 : \mathfrak{c} \rightarrow 2^\omega$ with the property that for every $\alpha, \beta \leq \gamma$

$$(\alpha, \beta, \gamma) \in A \text{ iff } (g(\alpha), g(\beta), h_1(\gamma)) \in F$$

Similarly let

$$D_2 = \{(\alpha, \beta, \gamma) : \alpha, \gamma \leq \beta < \mathfrak{c}\} \text{ and } D_3 = \{(\alpha, \beta, \gamma) : \gamma, \beta \leq \alpha < \mathfrak{c}\}$$

Now given any $A \subseteq D_2$ or $A \subseteq D_3$. and obtain h_2 , and h_3 with the analogous property.

Note that we may determine which case (D_i) in an F_σ way as follows. Let $k : \mathfrak{c} \rightarrow \omega^\omega$ be a scale, i.e., if $\alpha < \beta$ then $k(\alpha)(n) < k(\beta)(n)$ for all but

finitely many $n < \omega$. Such an object exists assuming Martin's axiom. We claim now that there exists an F_σ predicate H with the property that for all $A \subseteq \mathfrak{c} \times \mathfrak{c} \times \mathfrak{c}$ there exists $h : \mathfrak{c} \rightarrow 2^\omega$ such that

$$\forall \alpha, \beta, \gamma \ (\alpha, \beta, \gamma) \in A \text{ iff } (h(\alpha), h(\beta), h(\gamma)) \in H.$$

To see how to do this note that the function k can tell us which case we are in D_1 , D_2 , or D_3 . Then the function h codes up k and the h_1, h_2, h_3 .

Similarly, to the proof of Theorem 2.4, we get and $F : 2^\omega \times 2^\omega \times 2^\omega \rightarrow 2^\omega$ which is a level 2 Borel map which is universal.

QED

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